II B.Sc Maths - Differential Equations.

UNIT - II

Method of variation of parameters - II order differential equations with Constant coefficients for finding the P.I.'s of the form $e^{ax} \cdot V$, where V is $\sin(mx)$ or $\cos(mx)$ and x^n. Equations reducible to linear equations with constant coefficients - Cauchy's homogeneous linear equations - Legendre's linear equations.

Second order differential equations with Constant Coefficients

A second order linear D.E. is an equation of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \longrightarrow (1)$$

Here $a_2(x)$, $a_1(x)$, $a_0(x)$ & $b(x)$ are continuous functions of x & $a_2(x) \neq 0$.

When a_0, a_1, a_2 are constants, we say that the equation has constant coefficients. otherwise, it has variable coefficients.

$$(1) \Rightarrow \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x) \longrightarrow (2)$$

where $p(x) = \dfrac{a_1(x)}{a_2(x)}$, $q(x) = \dfrac{a_0(x)}{a_2(x)}$.

Now $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = 0 \longrightarrow (3)$ when R.H.S. $= 0$.

Here (2) is a non-homogeneous D.E. and (3) is a homogeneous D.E.

Note: (1) A linear D.E. of n^{th} order is of the form

$$a_0 \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 = F(x) \rightarrow (4)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ depend only on x or constants.

(2) If $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, then (4) is a linear D.E. with constant coefficients.

When $n=2$, $a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \rightarrow (5)$ is called a linear D.E. of order 2.

Definition:

A linear equation of the second order with constant coefficients is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X \longrightarrow (6)$$

where a, b, c are constants and X is a function of x.

Let $D = \frac{d}{dx} \Rightarrow Dy = \frac{dy}{dx}$, D is differential operator

$\therefore$ (5) $\Rightarrow (a_2 D^2 + a_1 D + a_0)y = 0 \rightarrow (6)$.

The solution of (5) has 2 parts.

1. Complementary function (C.F.)
2. Particular Integral (P.I.)

$\therefore$ Solution $Y = C.F. + P.I.$
$= Y_c + Y_p$.

Definition: The solution of the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \rightarrow (A)$ is called the Complementary function of the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X \rightarrow (B)$, where a, b, c are constants and X is a function of x. (i.e) when R.H.S. of (B) $= 0$).

Now, let us consider the II order L.D.E. as

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = ke^{ax} \cdot V \rightarrow (1)$$

where $V = \sin mx$ (or) $\cos mx$ (or) x^n.

Then Solution $y = C.F. + P.I.$

$D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$.

$\therefore (1) \Rightarrow (AD^2 + BD + C)y = ke^{ax} \cdot V \rightarrow (2)$.

Complementary function (C.F)

(2) $\Rightarrow$ Auxilliary equation is (A.E)

$Am^2 + Bm + C = 0 \rightarrow (3)$

Solving (3), we get 2 values of m.

Case (i) If the roots are distinct and real, say m_1, m_2, then $C.F. = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

Case (ii) Two roots are equal, say m, m, then

$C.F. = (C_1 x + C_2) e^{mx}$.

Case (iii) If the roots are imaginary of the form $\alpha \pm i\beta$, then

$C.F. = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

Type II

Form $(AD^2 + BD + c)y = 0$.

① Solve

ⓐ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

Solution:

$(D^2 - 3D + 2)y = 0$.

Auxiliary equation (A.E.) is

$m^2 - 3m + 2 = 0$.

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 1, 2$

C.F. is $y = Ae^{m_1 x} + Be^{m_2 x}$.

$\Rightarrow$ General solution is

$\boxed{y = Ae^{x} + Be^{2x}}$.

① ⓑ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

Solution:

$(D^2 + D - 2)y = 0$.

A.E. is $m^2 + m - 2 = 0$.

$\Rightarrow (m+2)(m-1) = 0$.

$\Rightarrow m = -2, 1$.

$\therefore$ C.F. is $y = c_1 e^{-2x} + c_2 e^{x}$.

① ⓒ $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$.

Solution: $(D^2 + 5D + 4)y = 0$.

A.E. is $m^2 + 5m + 4 = 0$.

$\Rightarrow (m-4)(m-1) = 0$.

$\Rightarrow m = 1, 4$

$y = c_1 e^{x} + c_2 e^{4x}$.

① ⓓ $(D^2 - 4D + 3)y = 0$.

Solution: A.E. is $m^2 - 4m + 3 = 0$.

$\Rightarrow (m-3)(m-1) = 0$.

$\Rightarrow m = 1, 3$.

$\therefore$ C.F. is $y = c_1 e^{x} + c_2 e^{3x}$

which is the general solution

① ⓔ $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$.

Solution: $D = \frac{d}{dt}, D^2 = \frac{d^2}{dt^2}$.

$\Rightarrow (D^2 + 6D + 9)x = 0$.

A.E. is $m^2 + 6m + 9 = 0$.

$\Rightarrow (m+3)^2 = 0$

$\Rightarrow m = -3, -3$.

$\therefore$ C.F. (or) G.S. is

$x = (c_1 + c_2 t)e^{-3t}$

① ⓕ $(D^2 - 6D + 9)y = 0$.

Solution: $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0 \Rightarrow m = 3, 3$

$\therefore$ Solution is

$y = (c_1 + c_2 x)e^{3x}$.

(or) $y = (c_1 x + c_2)e^{3x}$.

① ⓖ $(D^2 + D + 1)y = 0$

Solution: $m^2 + m + 1 = 0$.

$m = \frac{-1 \pm i\sqrt{3}}{2}$, $= \alpha \pm i\beta$.

$\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$

$\alpha = -1/2, \beta = \frac{\sqrt{3}}{2}$.

Solution is

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

$\Rightarrow y = e^{-1/2 x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}x}{2}\right)$.

1 (h) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0$.

Solution:

$(D^2 - 3D + 5)y = 0$.

A.E. is $m^2 - 3m + 5 = 0$.

$m = \frac{3 \pm \sqrt{9-20}}{2} = \frac{3 \pm \sqrt{11}\,i}{2} = \alpha \pm i\beta$.

$\alpha = \frac{3}{2}, \beta = \frac{\sqrt{11}}{2}$.

$\therefore y = e^{-3/2 x}\left(A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}x}{2}\right)$

1 (i) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$.

1 (j) $(3D^2 + D + 2)y = 0$.

[Ans. $\frac{-1 \pm i\sqrt{23}}{6}$]

1 (k) $(D^2 + D - 3)y = 0$.

1 (l) $(2D^2 + D - 3)y = 0$.

Particular Integral

Form: $(AD^2 + BD + C)y = F(x)$

where $F(x) = e^{ax}$.

Type II case (i)

$F(x) = e^{ax}$.

2 Solve

(a) $(D^2 + 5D + 6)y = e^x$.

Solution:

To find C.F.

A.E. is $(m^2 + 5m + 6) = 0$.

$(m+3)(m+2) = 0$.

$m = -2, -3$

C.F. is $y = Ae^{-2x} + Be^{-3x} \longrightarrow (1)$

P.I. $= \frac{1}{D^2 + 5D + 6}e^x$

$e^{ax} = e^x$

$\Rightarrow D = a = 1$.

Put $\boxed{D = 1}$

$\Rightarrow$ P.I. $= \frac{1}{1+5+6}e^x$

$= \frac{e^x}{12} \longrightarrow (2)$

$\therefore$ General solution is

$y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow \boxed{y = Ae^{-2x} + Be^{-3x} + \frac{e^x}{12}}$$

by (1) & (2)

2 (b) $(D^2 - 5D + 6)y = e^{4x}$

2 (c) $(D^2-4D+13)y = e^{2x}$.

Solution:

A.E. is $m^2-4m+13=0$.

$m = 2 \pm i3 = \alpha \pm i\beta$.

$C.F = e^{2x}(A\cos 3x + B\sin 3x)$

$P.I = \frac{1}{D^2-4D+13} e^{2x}$

$\boxed{D = a = 2}$

$= \frac{1}{4-8+13} e^{2x} = \frac{e^{2x}}{9}$

$\therefore y = C.F. + P.I.$

$\Rightarrow y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{e^{2x}}{9}$.

2 (d) $(D^3-3D^2+4D-2)y = e^{x}$.

Solution:

AE is $m^3-3m^2+4m-2=0$.

$m=1$ is a root.

1	1	-3	4	-2
	0	1	-2	2
	1	-2	2	0

$m^2-2m+2=0$

$m = 1 \pm i = \alpha \pm i\beta$.

C.F is

~~$y = Ae^x [A$~~

$\boxed{y = Ae^x + e^x(B\cos x + C\sin x)}$

$P.I. = \frac{1}{D^3-3D^2+4D-2} e^{x}$.

$\boxed{D = a = 1}$ $\quad (Dr=0)$

$= \frac{1}{1-3+4-2} e^x$

$= \frac{x}{3D^2-6D+4} e^x$.

$\boxed{D = a = 1}$

$= \frac{x}{3-6+4} e^x = xe^x$

$\boxed{P.I. = xe^x}$

$\therefore$ General solution is

$\boxed{y = C.F + P.I.}$

2 (e) $(D^2-3D+2)y = e^{5x}+2$

Solution:

A.E is $m^2-3m+2=0$

$m = 1, 2$

$C.F = Ae^x + Be^{2x}$.

$P.I. = \frac{1}{D^2-3D+2} e^{5x} + 2\frac{1}{D^2-3D+2} e^{0x}$.

$= \frac{e^{5x}}{25-15+2} + 2\cdot\frac{1}{2}$.

$= \frac{e^{5x}}{12} + 1$.

$\therefore$ Solution is

$y = C.F + P.I.$

$= Ae^x + Be^{2x} + \frac{e^{5x}}{12} + 1$.

2 (f) $(D^2 - D - 2)y = e^{2x} + e^{x}$.

Solution:

A.E. $m^2 - m - 2 = 0$.

$m = 2, -1$

C.F. $y = Ae^{2x} + Be^{-x}$.

$P.I. = \frac{1}{D^2 - D - 2} e^{2x} + \frac{1}{D^2 - D - 2} e^{x}$.

$\boxed{D = a = 2}$ $\boxed{D = a = 1}$

$= \frac{1}{4 - 2 - 2} e^{2x} + \frac{1}{1 - 1 - 2} e^{x}$.

$\boxed{Dr = 0}$

$= \frac{x}{2D - 1} e^{2x} + \frac{1}{-2} e^{x}$.

$= \frac{x}{4 - 1} e^{2x} - \frac{1}{2} e^{x}$

$\therefore P.I. = \frac{xe^{2x}}{3} - \frac{e^{x}}{2}$.

$y = C.F. + P.I.$ is G. solution

2 (g) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$.

Solution: $(D^2 + 4D + 4)y = e^{-2x}$

To find C.F.

A.E. $m^2 + 4m + 4 = 0$

$(m + 2)^2 = 0 \Rightarrow m = -2, -2$

$\boxed{C.F. = (Ax + B)e^{-2x}}$

To find P.I.

$P.I. = \frac{1}{D^2 + 4D + 4} e^{-2x}$.

$\boxed{D = a = -2}$

$= \frac{1}{4 - 8 + 4} e^{-2x} \quad [Dr = 0]$

$= \frac{x}{2D + 4} e^{-2x} = \frac{x}{-4 + 4} e^{-2x} \quad \boxed{Dr = 0}$

$= \frac{x^2}{2} e^{-2x}$.

$\therefore y = C.F. + P.I.$

2 (h) $(D^2 + 2D + 1)y = e^{-x} + 3$.

Solution: $(m^2 + 2m + 1) = 0$.

$C.F. = (Ax + B)e^{-x}$

$P.I. = \frac{1}{D^2 + 2D + 1} e^{-x} + \frac{3 e^{0x}}{D^2 + 2D + 1}$

$\boxed{D = a = -1}$ $\boxed{D = a = 0}$

$(Dr = 0)$

$= \frac{x}{2D + 2} e^{-x} + 3$.

$\boxed{Dr = 0}$

$= \frac{x^2}{2} e^{-x} + 3$.

$y = C.F. + P.I.$

2 (i) $(D^2 + 5D - 6)y = 3e^{x}$.

[Ans. $m = 1, -6$, $P.I. = \frac{3}{7} x e^{x}$]

2 (j) $(D^2 - 6D + 9)y = e^{3x}$.

[$m = 3, 3$ $P.I. = \frac{x^2}{2} e^{3x}$.]

2 (k) $(D^2 + 5D + 6)y = 3e^{2x}$.

[$m = -3, -2$, $P.I. = \frac{3}{20} e^{2x}$].

(2) (l) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=5e^{-x}$.

$[m=-3,-3,\ P.I.=\frac{5}{4}e^{-x}]$

(2) (m) $(D^2+5D+6)y=2e^{-3x}$

$[m=-3,-2,\ P.I.=-2xe^{-3x}]$.

(2) (n) $(D^2+4D+4)y=\frac{3}{2}e^{-2x}$.

$[m=-2,-2,\ P.I.=\frac{3}{2}\frac{x^2}{2}e^{-2x}]$

(2) (o) PI of $(\cancel{D^2+5D+6)y=e^{x}}$

Solution: $(D+2)(D-1)^2y=e^{-2x}$.

$$P.I.=\frac{1}{(D+2)(D-1)^2}e^{-2x}.$$

$$=\frac{1}{(D+2)}\left[\frac{1}{(D-1)^2}e^{-2x}\right].$$

$$=\frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right].$$

$$\cancel{x\cdot\frac{1}{1}\cdot e^{-2x}}$$

$$=\frac{1}{9}.$$

$$=\frac{x}{9}e^{-2x}$$

$$=\frac{1}{D+2}\cdot\frac{1}{(-2-1)^2}e^{-2x}$$

$$=\frac{1}{9}\cdot\frac{1}{D+2}e^{-2x}\quad\left[D=-2\Rightarrow Dr=0\right]$$

$$=\frac{1}{9}\cdot x\cdot\frac{1}{1}\cdot e^{-2x}.$$

$$=\frac{x}{9}e^{-2x}.$$

(2) (p)

$(3D^2+D-14)y=13e^{2x}$.

Solution:

A.E. $3m^2+m-14=0$.

$m=2,-7/3$

$y=Ae^{2x}+Be^{-\frac{7}{3}x}$ is C.F.

$$P.I.=\frac{1}{(D-2)(3D+7)}13e^{2x}.\quad\boxed{D=a=2}$$

$$=13.\frac{1}{(D-2)(6+7)}e^{2x}$$

$$=\frac{1}{D-2}e^{2x}\quad D=a=2\Rightarrow Dr=0.$$

$$=\frac{x}{1}e^{2x}$$

P.I. $=xe^{2x}$.

$\therefore$ C.F. + P.I. is G. solution

$y=$

Type III

$(AD^2+BD+C)y = k\cos ax$ (or) $k\sin ax$

$D^2 = -(a^2)$

③ Solve

ⓐ $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y =$ (i) $4\cos 3x$. (ii) $4\cos^2 x$

(i)

Solution: Given D.E. is

$(D^2+3D+2)y = 4\cos 3x \to (1)$

To find C.F

Consider $(D^2+3D+2)y=0$

A.E. is $m^2+3m+2=0$.

$(m+1)(m+2)=0$

$\Rightarrow m=-1,-2$.

$\therefore$ C.F. is $y = Ae^{-x}+Be^{-2x}$.

To find P.I.

(i) $P.I. = \frac{1}{D^2+3D+2}\, 4\cos 3x$. $\quad D^2=-(a^2)=-9$

$= 4.\frac{1}{-9+3D+2}\cos 3x$.

$= 4\cdot\frac{1}{3D-7}\cos 3x$

$= 4\frac{3D+7}{(3D-7)(3D+7)}\cos 3x$.

$= 4\frac{3D+7}{9D^2-49}\cos 3x$.

$= 4\frac{3D+7}{9(-9)-49}\cos 3x$

$= 4\frac{3D+7}{-130}\cos 3x$.

$= -\frac{4}{130}\left[3D(\cos 3x)+7\cos 3x\right]$.

$= -\frac{4}{130}\left[3.3.(-\sin 3x)+7\cos 3x\right]$

$= -\frac{4}{130}\left[7\cos 3x - 9\sin 3x\right]$.

General solution is

$y = C.F. + P.I.$

(ii) Ans. $P.I. = 1 + \frac{3\sin 2x - \cos 2x}{10}$

ⓐ (ii) $(D^2+3D+2)y = \sin 3x$.

Ans. $P.I. = -\frac{(9\cos 3x+7\sin 3x)}{130}$

(iii) ⓐ $(D^2+3D+2)y = \cos x$

Ans $P.I. = \frac{3\sin x+\cos x}{10}$

ⓑ $(D^2-3D+2)y = \sin 3x$

Ans. $P.I. = -\frac{1}{130}\left[7\sin 3x - 9\cos 3x\right]$.

ⓒ $(D^2-4D+4)y = e^{2x}+\cos 2x$.

Ans. $m=2,2$ $P.I. = \frac{x^2e^{2x}}{2} - \frac{\sin 2x}{8}$

ⓓ $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = \cosh 2x$. $\quad \cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$

Ans. $m=-1,-1$

$P.I._1 = \frac{e^{2x}}{18}$, $P.I._2 = \frac{e^{-2x}}{2}$.

ⓔ (i) $(D^2-4D+3)y = \cos 2x$

Ans: $m=-3,-1$

$P.I. = -\frac{1}{65}(8\sin 2x+\cos 2x)$

ⓔ (ii) $(D^2-4D+3)y = 5\sin 2x$

Ans $P.I. = \frac{8\cos 2x - \sin 2x}{13}$

(3f) $(D^2+6D+8)y = e^{-2x} + \cos^2 x.$

Ans. $m = -4, -2$

$P.I_1 = \frac{x}{2} e^{-2x}.$

$P.I._2 = \frac{1}{D^2+6D+8} \cdot \left(\frac{1+\cos 2x}{2}\right).$

$= \frac{1}{16} - \frac{1}{320}(12\sin 2x - 4\cos 2x)$

(3g) $(D^2+16)y = 2e^{-3x} + \cos 4x.$

(i) Solution:

C.F $(D^2+16)y = 0.$

A.E. is $m^2+16 = 0.$

$m = \pm 4i = \alpha \pm i\beta \Rightarrow \alpha = 0, \beta = 4$

C.F. is $y = A\cos 4x + B\sin 4x.$

$P.I_2 = \frac{1}{D^2+16} \cos 4x.$ $D^2 = -a^2 = -16$

$\boxed{Dr = 0}$

$= \frac{x}{2D} \cos 4x.$

$= \frac{x}{2} \int \cos 4x \, dx.$

$= \frac{x}{2} \cdot \frac{\sin 4x}{4}.$

$P.I._1 = \frac{1}{D^2+16} 2e^{-3x}$ $\boxed{D = -3}$

$= 2 \cdot \frac{1}{9+16} e^{-3x}.$

$= \frac{2}{25} e^{-3x}.$

$y = C.F + P.I_1 + P.I_2$

is the general solution.

(ii) P.I of $(D^2+16)y = \cos 4x.$

Alternate method

$P.I. = \frac{1}{D^2+16} \cos 4x.$

$= \frac{1}{D^2+16}$ Real part of $e^{4ix}.$ (2)

$= \text{Real part of } \frac{1}{(D+4i)(D-4i)} e^{4ix}.$

$= R.P. \text{ of } \frac{1}{8i(D-4i)} e^{4ix}.$ $\boxed{D = 4i}$

$= R.P. \text{ of } \frac{1}{8i} \cdot \frac{1}{D-4i} e^{4ix}$

$\boxed{D = 4i \Rightarrow Dr = 0}$

$= R.P. \text{ of } \frac{1}{8i} \cdot x e^{4ix}.$

$= R.P. \text{ of } \frac{1}{8i}(\cos 4x + i\sin 4x).$

$= R.P. \text{ of } -\frac{ix}{8}(\cos 4x + i\sin 4x)$

$= \frac{x}{8} \sin 4x.$

$y = C.F. + P.I.$

(3h) (i) $(D^2+4)y = 5\sin 2x.$

Solution: C.F

Consider $(D^2+4)y = 0.$

A.E. is $m^2+4 = 0.$

$m = \pm 2i$

C.F. is $y = A\cos 2x + B\sin 2x.$

$P.I. = \frac{1}{D^2+4} 5\sin 2x.$ $D^2 = -4$

$= 5.$ $\boxed{Dr = 0}$.

$= 5 \frac{1}{2D} \cdot x \sin 2x.$

$= \frac{5x}{2}\left[-\cos 2x\right]$

$= -\frac{5x\cos 2x}{2}$

$y = C.F + P.I.$

(ii) $(D^2+4)y = \sin 3x.$

3 (i) $(D^2+9)y = 4\cos 3x$.

Ans- $m = \pm 3i$

$P.I. = \dfrac{2x\sin 3x}{3}$.

3 (j) $(D^4+D^3+D^2)y = \cos x$.

Ans- $m = 0, 0, \dfrac{-1 \pm i\sqrt{3}}{2}$.

$P.I. = -\dfrac{1}{D}\cos x = -\sin x$.

3 (k) $(D^4-2D^3+D^2)y = \cos x$.

Ans- $m = 0, 0, 1, 1$.

$C.F = (Ax+B)(Cx+D)e^{-x}$.

$P.I. = \dfrac{1}{2D}\cos x = \dfrac{1}{2}\sin x$.

3 (l) $(D^2-4D+3)y = \sin 3x\cos 2x$.

Ans- $m = 1, 3$.

$\sin 3x\cos 2x = \dfrac{\sin 5x + \sin x}{2}$

$P.I. = \dfrac{1}{2}\left[\dfrac{10\cos 5x - 11\sin 5x}{442} + \dfrac{2\cos x + \sin x}{10}\right]$.

Type IV

$(AD^2+BD+C)y =$ polynomial function of x.

4 Solve

(a) $(D^2+5D+4)y = x^2+2x-1$.

Solution:

C.F. $(D^2+5D+4)y = 0$.

$m = -1, -4$

$C.F = y = Ae^{-x} + Be^{-4x}$.

$P.I. = \dfrac{1}{D^2+5D+4}(x^2+2x+1)$.

$= \dfrac{1}{4\left[1+\left(\frac{D^2+5D}{4}\right)\right]}(x^2+2x+1)$

$= \dfrac{1}{4}\left[1+\left(\dfrac{D^2+5D}{4}\right)\right]^{-1}(x^2+2x+1)$.

$= \dfrac{1}{4}\left[1-\left(\dfrac{D^2+5D}{4}\right)+\left(\dfrac{D^2+5D}{4}\right)^2 - \cdots\right](x^2+2x+1)$

$D(x^2) = 2x$

$D^2(x^2) = 2$.

$= \dfrac{1}{4}\left[1-\dfrac{D^2}{4}-\dfrac{5D}{4}+\dfrac{25D^2}{16}\right](x^2+2x+1)$.

$= \dfrac{1}{4}\left[(x^2+2x+1)-\left(\dfrac{2}{4}\right)-\dfrac{5}{4}(2x+2)+\dfrac{25}{16}\right]$.

$= \dfrac{1}{4}\left[x^2+2x+1-\dfrac{5}{2}x-\dfrac{5}{2}-\dfrac{1}{2}+\dfrac{25}{16}\right]$

$= \dfrac{1}{4}\left[16x^2+32x+16-40x-40-8+25\right]$.

$= \dfrac{1}{64}\left[16x^2-8x-7\right)$.

4 (b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2+3$.

Solution:

$(D^2-5D+6)y = 0$.

$m = 3, 2$

C.F. is $y = Ae^{3x} + Be^{2x}$.

$P.I. = \frac{1}{D^2-5D+6} \cdot (x^2+3)$

$= \frac{1}{6\left[1+\frac{D^2-5D}{6}\right]} (x^2+3)$.

$= \frac{1}{6}\left[1+\left(\frac{D^2-5D}{6}\right)\right]^{-1} (x^2+3)$.

$= \frac{1}{6}\left[1 - \frac{D^2-5D}{6} + \left(\frac{D^2-5D}{6}\right)^2\right](x^2+3)$.

$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25D^2}{36}\right](x^2+3)$.

$= \frac{1}{6}\left[x^2+3 - \frac{2}{6} + \frac{10x}{6} + \frac{25}{36}\right]$.

$= \frac{1}{6}\left[\frac{36x^2+108-12+60x+25}{36}\right]$.

$= \frac{1}{216}\left[36x^2+60x+121\right]$.

4 (c) P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$.

Solution:

$P.I. = \frac{1}{D^2+D} (x^2+2x+4)$.

$= \frac{1}{D(D+1)} (x^2+2x+4)$.

$= \frac{1}{D} (1+D)^{-1} (x^2+2x+4)$.

$= \frac{1}{D} (1-D+D^2-\cdots)(x^2+2x+4)$.

$= \frac{1}{D}\left[x^2+2x+4-(2x+2)+2\right]$.

$= \frac{1}{D}\left[x^2+4\right] = \int (x^2+4)\,dx$.

$= \frac{x^3}{3} + 4x$.

(d) $(D^2+5D+4)y = x^2+7x+9$.

Ans. $m = -1, -4$.

$P.I._1 = \frac{x^2}{4} - \frac{5}{8}x + \frac{21}{32}$.

$P.I._2 = \frac{7}{4}x - \frac{35}{16}$

$P.I._3 = \frac{1}{32}(8x^2+36x+23)$.

(e) $(D^2+D-6)y = x$.

Ans. $m = -3, 2$

$P.I. = -\frac{1}{36}(6x+1)$

(f) $(D-2)^2y = 8(e^{2x}+\sin 2x+x^2)$.

Ans. $m = 2, 2$.

$P.I._1 = \frac{e^{2x}.x^2}{2}$, $P.I._2 = \frac{\cos 2x}{8}$

$P.I._3 = \frac{2x^2+4x+15}{8}$

Type V

$(AD^2+BD+C)y = ke^{ax}\cdot V.$

where $V = \sin mx$ (or) $\cos mx$ (or) x^n

(5) Solve

(a)(i) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y =$ (i) $3e^{2x}\sin 4x.$ (ii) $e^{-2x}\sin 2x$

Solution:

$(D^2+5D+6)y = 0.$

C.F. $m = -3, -2$

C.F. is $y = Ae^{-3x} + Be^{-2x}.$

$P.I. = \frac{1}{D^2+5D+6}\cdot 3e^{2x}\sin 4x. \rightarrow (1)$

$e^{ax}\sin bx = e^{2x}\sin 4x$

$\Rightarrow a=2,\ b=4.$

$D = D+a = D+2.$

$\therefore P.I. = 3e^{2x}\frac{1}{(D+2)^2+5(D+2)+6}\sin 4x.$

$= 3e^{2x}\frac{1}{D^2+9D+20}\sin 4x.$

$D^2 = -b^2 = -16$ if $Dr \neq 0.$

$= 3e^{2x}\cdot\frac{1}{-16+9D+20}\sin 4x.$

$= 3e^{2x}\frac{1}{9D+4}\sin 4x.$

$= 3e^{2x}\frac{9D-4}{(9D+4)(9D-4)}\sin 4x.$

$= 3e^{2x}\cdot\frac{(9D-4)\sin 4x}{81D^2-16}$

$= 3e^{2x}\frac{[9D(\sin 4x) - 4\sin 4x]}{81(-16)-16}$

$= 3e^{2x}\frac{[9.4\cos 4x - 4\sin 4x]}{-1312}$

$= \frac{3e^{2x}[\sin 4x - 9\cos 4x]}{328}.$

(ii) $P.I. = e^{-2x}\frac{(2\cos 2x + 4\sin 2x)}{-20}$

(b) $(D^2+4D+3)y =$ (i) $e^{-x}\sin x$ (ii) $(x^2+2)e^{3x}.$ (iii) $e^x\cos 2x - \cos 3x$

Solution:

C.F $= c_1e^{-3x} + c_2e^{-x}.$

P.I. (i) $P.I. = \frac{1}{D^2+4D+3}e^{-x}\sin x.$

$\boxed{D = D-1}$

$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x.$

$= e^{-x}\cdot\frac{1}{D^2+2D}\sin x. \quad \boxed{D^2=-1}$

$= e^{-x}\cdot\frac{1}{-1+2D}\sin x.$

$= e^{-x}\cdot\frac{2D+1}{(2D-1)(2D+1)}\sin x.$

$= e^{-x}\cdot\frac{2D+1}{4D^2-1}\sin x.$

$= e^{-x}\frac{[2D(\sin x) + \sin x]}{4(-1)-1}$

$= -\frac{e^{-x}}{5}[2\cos x + \sin x.$

(b)(ii) $(x^2+2)e^{3x}$

$$P.I. = \frac{1}{D^2+4D+3} (x^2+2)e^{3x} \quad \boxed{D=D+3}$$

$$= e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}(x^2+2)$$

$$= e^{3x}\cdot\frac{1}{D^2+10D+24}(x^2+2)$$

$$= \frac{e^{3x}}{24}\left[1+\left(\frac{D^2+10D}{24}\right)\right]^{-1}(x^2+2)$$

$$= \frac{e^{3x}}{24}\left[1-\frac{D^2+10D}{24}+\left(\frac{D^2+10D}{24}\right)^2\right](x^2+2)$$

$$= \frac{e^{3x}}{24}\left[(x^2+2)-\frac{D^2}{24}(x^2+2)-\frac{10}{24}D(x^2+2) + \frac{100D^2}{576}(x^2+2)\right].$$

$$= \frac{e^{3x}}{24}\left[x^2+2-\frac{2}{24}-\frac{20x}{24}+\frac{200}{576}\right].$$

$$= \frac{e^{3x}}{24}\left[x^2+2-\frac{1}{12}-\frac{5}{6}x+\frac{25}{72}\right].$$

$$= \frac{e^{3x}}{24}\left[\frac{72x^2-60x+163}{72}\right].$$

(b)(iii) $e^x\cos 2x - \cos 3x$

$$P.I._1 = -\frac{1}{160}e^x(12\sin 2x - 4\cos 2x)$$

$$P.I._2 = \frac{1}{30}(2\sin 3x - \cos 3x)$$

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(5) (c) $y'' - 2y' + 2y = e^x\cos x$

Solution:

C.F. $(D^2-2D+2)y=0.$

$m = 1\pm i$

C.F is $y = e^x(A\cos x + B\sin x)$

$$P.I. = \frac{1}{D^2-2D+2}e^x\cos x$$

$$= e^x\cdot\frac{1}{(D+1)^2-2(D+1)+2}\cos x.$$

$$= e^x\cdot\frac{1}{D^2+1}\cos x.$$

$D^2=-1 \Rightarrow Dr = \infty.$

$$= e^x\cdot\frac{x}{2D}\cos x.$$

$$= e^x\cdot\frac{x}{2}\int\cos x\,dx$$

$$= e^x\,\frac{x}{2}\sin x.$$

$y = C.F. + P.I.$

---

(5)(d) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y =$

(i) $e^{-x}\sin 2x$

(ii) $\frac{e^{-2x}}{x^2}$.

Solution:

C.F $(D^2+4D+4)y=0$

$m=-2,-2$

C.F. $= y = (Ax+B)e^{-2x}$.

P.I. (i) $e^{-x}\sin 2x$.

$$P.I. = \frac{1}{D^2+4D+4}e^{-x}\sin 2x.$$

$$= e^{-x}\cdot\frac{1}{(D-1)^2+4(D-1)+4}\sin 2x$$

$$= e^{-x}\cdot\frac{1}{D^2+2D+1}\sin 2x.$$

$D^2=-4.$

$= e^{-x} \cdot \frac{1}{-4+2D+1} \sin 2x$

$= e^{-x} \cdot \frac{2D+3}{(2D-3)(2D+3)} \sin 2x$

$\cancel{= e^{-x}\left[2D(\sin 2x) + 3\sin 2x\right]}$

$= e^{-x} \cdot \frac{2D+3}{4D^2-9} \sin 2x$

$= e^{-x} \frac{2D+3}{4(-4)-9} \sin 2x$

$= -\frac{e^{-x}}{25}\left[2D(\sin 2x) + 3\sin 2x\right]$

$= \frac{-e^{-x}}{25}\left[4\cos 2x + 3\sin 2x\right]$

4(ii) $\frac{e^{-2x}}{x^2}$; $e^{-2x} \cdot x^{-2}$

$P.I. = \frac{1}{D^2+4D+4}$.

$= e^{-2x} \frac{1}{(D-2)^2+4(D-2)+4} x^{-2}$

$= e^{-2x} \frac{1}{D^2-4D+4+4D-8+4} x^{-2}$

$= e^{-2x} \cdot \frac{1}{D^2} \cdot x^{-2}$

$= e^{-2x} \frac{1}{D}\int x^{-2}\,dx$

$= e^{-2x} \frac{1}{D}\left(\frac{x^{-1}}{-1}\right)$

$= -e^{-2x} \int \frac{dx}{x}$

$= -e^{-2x} \log x$

5 (a) $(D^2-2D+1)y = x^2 e^{3x}$

Solution:

C.F $m = 1, 1$

$P.I = \frac{1}{D^2-2D+1} e^{3x} \cdot x^2$

$= e^{3x} \cdot \frac{1}{(D+3)^2-2(D+3)+1} x^2$

$= e^{3x} \cdot \frac{1}{D^2+4D+4} x^2$

$= \frac{e^{3x}}{4} \cdot \frac{1}{1+\left(\frac{D^2+4D}{4}\right)} x^2$

$= \frac{e^{3x}}{4}\left[1+\left(\frac{D^2+4D}{4}\right)\right]^{-1}(x^2)$

$= \frac{e^{3x}}{4}\left[1 - \frac{D^2+4D}{4} + \left(\frac{D^2+4D}{4}\right)^2\right](x^2)$

$= \frac{e^{3x}}{4}\left[x^2 - \frac{2}{4} - 2x + \frac{16}{16}\cdot 2\right]$

$= \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right)$

5 (f) $(D^2+9)y = (x^2+1)e^{3x}$

Solution: $m = \pm 3i$

$P.I = \frac{1}{D^2+9} e^{3x}(x^2+1)$

$= e^{3x} \cdot \frac{1}{(D+3)^2+9}(x^2+1)$

$= e^{3x} \cdot \frac{1}{D^2+6D+18}(x^2+1)$

$= \frac{e^{3x}}{18}\left[1+\left(\frac{D^2+6D}{18}\right)\right]^{-1}(x^2+1)$

$= \frac{e^{3x}}{18}\left[(x^2+1) - \frac{D^2+6D}{18}(x^2+1) + \left(\frac{D^2+6D}{18}\right)^2(x^2+1)\right]$

$$= \frac{e^{3x}}{18}\left[x^2+1 - \frac{2}{18} - \frac{12x}{18} + \frac{36\cdot(2)}{18\cdot 18}\right].$$

$$= \frac{e^{3x}}{18}\left[x^2+1-\frac{1}{9}-\frac{2x}{3}+\frac{2}{9}\right].$$

$$= \frac{e^{3x}}{18}\left[\frac{9x^2-6x+10}{9}\right].$$

5 (g) $(D^2-4D+3)y = $ (i) $e^x \cos 2x$
(ii) $e^{-x}\sin x$.

Solution:

C.F. $m = 3, 1$.

(i) $$P.I. = \frac{1}{D^2-4D+3} e^x \cos 2x.$$

$$= e^x \cdot \frac{1}{(D+1)^2-4(D+1)+3} \cos 2x.$$

$$= e^x \cdot \frac{1}{D^2-2D} \cos 2x.$$

$$= e^x \cdot \frac{1}{D(D-2)} \cos 2x.$$

$$= -\frac{e^x}{2}\cdot\frac{1}{D}\left(1-\frac{D}{2}\right)^{-1}\cos 2x$$

$$= -\frac{e^x}{2}\cdot\frac{1}{D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right]\cos 2x$$

$$= -\frac{e^x}{2}\frac{1}{D}\left[\cos 2x - \frac{2\sin 2x}{2} + 4\cos 2x\right].$$

$$= -\frac{e^x}{2}\cdot\frac{1}{D}\left[5\cos 2x - \sin 2x\right]$$

$$= \frac{-e^x}{2}\left[\frac{5}{2}\sin 2x - \frac{\cos 2x}{2}\right]$$

$$= \frac{e^x}{4}\left[\cos 2x - 5\sin 2x\right].$$

(g) (ii) $e^{-x}\sin x$. H.W.

Ans. $\frac{e^{-x}}{85}(7\sin x + 6\cos x)$

(h) $(D^2-2D+5)y = e^x \sin 2x$.

Ans. $m = 1 \pm 2i$

$P.I. = -e^x \cdot \frac{x}{4}\cos 2x$

(i) $(D^2-3D+2)y = x^2 e^{3x}$.

Ans. $m = 2, 1$.

$P.I. = \frac{e^{3x}}{2}\left(x^2-3x+\frac{7}{2}\right)$.

(j) $(D^2+4D+3)y = xe^{3x}$.

Ans. $m = -3, -1$.

$P.I. = \frac{e^{3x}}{24}\left(x - \frac{5}{12}\right)$.

(k) $(D^2-1)y = x^2 e^x$.

Solution: C.F $= m = \pm 1$.

$$P.I. = \frac{1}{D^2-1} x^2 e^x.$$

$$= e^x \cdot \frac{1}{(D+1)^2-1} x^2 = \frac{e^x}{D^2+2D} x^2.$$

$$= e^x \cdot \frac{1}{D}\cdot\frac{1}{D+2} x^2 = e^x \cdot \frac{1}{D}\cdot\frac{1}{2\left(1+\frac{D}{2}\right)} x^2.$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(x^2).$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}\right)(x^2)$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\left(x^2 - \frac{2x}{2} + \frac{2}{4}\right).$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\left(x^2 - x + 1/2\right).$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\left(x^2 - x + 1/2\right)$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right].$$

$$= \frac{e^x}{12}\left(2x^3 - 3x^2 + 3x\right).$$

(5)(l) $(D^2-4)y = x\sinh x$.

Solution: C.F $m = \pm 2$.

$$P.I. = \frac{1}{D^2-4}\, x\sinh x.$$

$$= \frac{1}{D^2-4} \cdot x\left(\frac{e^x - e^{-x}}{2}\right).$$

$$= \frac{1}{2} \cdot \frac{1}{D^2-4} xe^x - \frac{1}{2}\cdot\frac{1}{D^2-4} xe^{-x}.$$

$$= \frac{1}{2} \cdot e^x \frac{1}{(D+1)^2-4} x - \frac{1}{2}\cdot\frac{1}{(D-1)^2-4} x.$$

$$= \frac{1}{2} e^x \cdot \frac{1}{D^2+2D-3} x - \frac{1}{2}\cdot\frac{1}{D^2-2D-3} x.$$

$$= \frac{-e^x}{6}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1}(x) + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1}(x).$$

$$= \frac{-e^x}{6}\left[1-\frac{2D}{3}\right](x) + \frac{e^{-x}}{6}\left[1+\frac{2D}{3}\right](x).$$

$$= -\frac{e^x}{6}\left[x-\frac{2}{3}\right] + \frac{e^{-x}}{6}\left[x+\frac{2}{3}\right].$$

$$= -\frac{xe^x}{6} + \frac{e^x}{9} + \frac{xe^{-x}}{6} + \frac{e^{-x}}{9}.$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) + \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= -\frac{x}{3}\sinh x + \frac{2}{9}\cosh x. //$$

5 m) P.I of $(D^2-2D+4)y = e^x\cos x$.

Ans: $= \dfrac{e^x \cos x}{2}$

5) n) $(D^2+2D+5)y = xe^x$.

Solution:

$m = -1 \pm 2i$

C.F $= e^{-x}(A\cos 2x + B\sin 2x)$.

$$P.I. = \frac{1}{D^2+2D+5} xe^x.$$

$$= e^x \cdot \frac{1}{(D+1)^2+2(D+1)+5} x.$$

$$= e^x \cdot \frac{1}{D^2+4D+8} x.$$

$$= \frac{e^x}{8}\left[1+\frac{D^2+4D}{8}\right]^{-1} x.$$

$$= \frac{e^x}{8}\left[1-\frac{4D}{8}\right](x).$$

$$= \frac{e^x}{8}\left(x-\frac{1}{2}\right).$$

$y = C.F + P.I$ is general solution.

5) o) solve $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Soln: Ans. C.F $c_1e^{2x} + c_2e^{-2x}$.

$$P.I. = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$